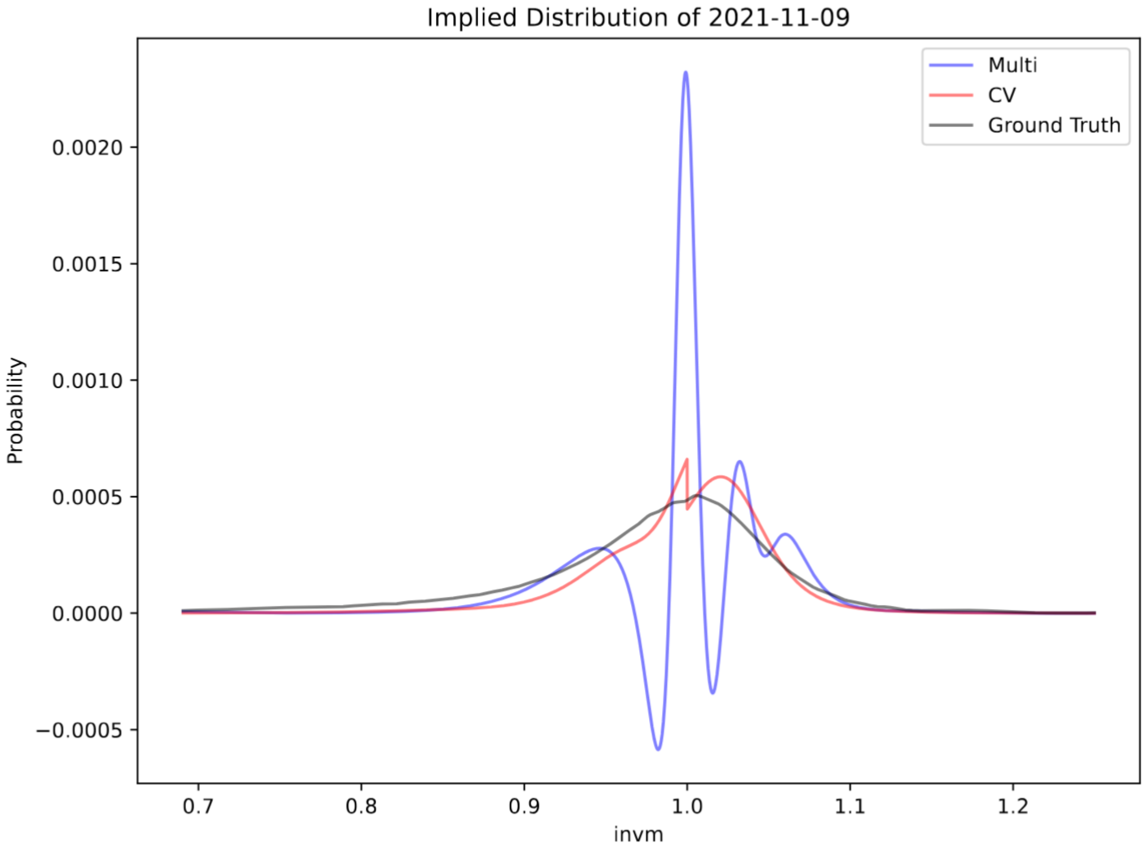
CV Model



visulization.py
refer to Hull J.C. - Options, Futures, and Other Derivatives p. 447 Addendix, DETERMINING IMPLIED RISK-NEUTRAL DISTRIBUTIONS FROM VOLATILITY SMILES

Deep Learning of Transition Probability Densities for Stochastic Asset Models with Applications in Option Pricing

We know that the solution of the **Fokker-Planck (Kolmogorov forward) equation** is the transition probability density. Its initial condition is a Dirac delta function, which has zero value everywhere except at one point where it is infinite. The presence of the delta function in the initial condition makes it impractical to use deep learning methods for the Fokker-Plank equation.

Derivation and Context

The Fokker-Planck equation can be derived from the Itô calculus, particularly from the stochastic differential equation (SDE) that describes the motion of the particle:

$$dx_t = A(x_t, t)dt + \sqrt{2B(x_t, t)}\, dW_t$$

Here, dW_t represents a Wiener process or Brownian motion, and $A(x_t, t)$ and $B(x_t, t)$ correspond to the drift and diffusion terms, respectively.

General Form

The general form of the Fokker-Planck equation is:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x,t)P(x,t)] + \frac{\partial^2}{\partial x^2}[B(x,t)P(x,t)]$$

Where:

- $P(x,t)$ is the probability density function of the stochastic variable x at time t.
- $A(x,t)$ is the drift coefficient, representing deterministic forces acting on the particle.
- $B(x,t)$ is the diffusion coefficient, representing random forces (stochastic noise).

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$$dx_t = A(x_t, t)dt + \sqrt{2B(x_t, t)}\, dW_t$$

The general form of the Fokker-Planck equation is:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x,t)P(x,t)] + \frac{\partial^2}{\partial x^2}[B(x,t)P(x,t)]$$

$$dX = -\frac{1}{2}\sigma^2(s, e^{X(s)})ds + \sigma(t, e^{X(s)})dW, \; X(t) = x_0 = \ln S_0. \tag{5}$$

$$\frac{\partial}{\partial s}p(s,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(s,e^y)p(s,y)] + \frac{1}{2}\frac{\partial}{\partial y}[\sigma^2(s,e^y)p(s,y)], \; s > 0, \; y \in \mathbb{R}, \tag{7}$$

with initial condition for a fixed x

$$p(0,y) = \delta(y - x), \; y \in \mathbb{R}, \tag{8}$$

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We know that the solution of the **Fokker-Planck (Kolmogorov forward) equation** is the transition probability density. Its initial condition is a **Dirac delta function**, which has zero value everywhere except at one point where it is infinite. The presence of the delta function in the initial condition makes it impractical to use deep learning methods for the Fokker-Plank equation.

$$\frac{\partial}{\partial s}p(s,y) = \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(s,e^y)p(s,y)] + \frac{1}{2}\frac{\partial}{\partial y}[\sigma^2(s,e^y)p(s,y)],\ s>0,\ y\in\mathbb{R}, \quad (7)$$

with initial condition for a fixed x

$$p(0,y) = \delta(y-x),\ y\in\mathbb{R}, \quad (8)$$

$$\delta(y-x) = \begin{cases} +\infty, & y = x, \\ 0, & y \neq x, \end{cases} \quad (9)$$

Because the function is infinite at the point y = x, it is difficult to solve the problem (7)–(8) numerically.

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Backward Kolmogorov Equation for Cumulative Distribution Function.

Context and Definition

Consider a stochastic process X_t governed by the stochastic differential equation (SDE):

$$dX_t = \mu(X_t, t)\,dt + \sigma(X_t, t)\,dW_t,$$

where:

- $\mu(X_t, t)$ is the drift term.
- $\sigma(X_t, t)$ is the diffusion term.
- W_t is a standard Wiener process.

Form of the Backward Kolmogorov Equation for the CDF

The Backward Kolmogorov Equation for the CDF $F(x, t)$ is given by:

$$\frac{\partial F(x,t)}{\partial t} + \mu(x,t)\frac{\partial F(x,t)}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 F(x,t)}{\partial x^2} = 0.$$

This equation represents how the probability that the process X_t has reached or exceeded a level x evolves backward in time.

$$dX = -\frac{1}{2}\sigma^2(s, e^{X(s)})ds + \sigma(t, e^{X(s)})dW,\ X(t) = x_0 = \ln S_0. \quad (5)$$

↓

$$\frac{\partial}{\partial t}C(t,x) + \frac{1}{2}\sigma^2(t,e^x)\frac{\partial^2}{\partial x^2}C(t,x) - \frac{1}{2}\sigma^2(t,e^x)\frac{\partial}{\partial x}C(t,x) = 0,\ t \in [0,T),\ x \in \mathbb{R}, \quad (11)$$

with the terminal condition for a fixed y

$$C(T,x) = \mathbb{1}(x \le y) = \begin{cases} 1, & x \le y, \\ 0, & x > y, \end{cases} \quad x \in \mathbb{R}, \quad (12)$$

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$$dX(u) = \beta\big(u, X(u)\big)du + \gamma\big(u, X(u)\big)dW(u).$$

Forward Kolmogorov Equation.

$$\frac{\partial}{\partial T}p(t,T,x,y) = -\frac{\partial}{\partial y}\big(\beta(t,y)p(t,T,x,y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T,y)p(t,T,x,y)\big).$$

t and x were held constant and the variables were T and y

Backward Kolmogorov Equation.

$$-p_t(t,T,x,y) = \beta(t,x)p_x(t,T,x,y) + \frac{1}{2}\gamma^2(t,x)p_{xx}(t,T,x,y).$$

T and y were held constant and the variables were t and x

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Exercise 6.9 (Kolmogorov forward equation).

(Also called the *Fokker-Planck equation*). We begin with the same stochastic differential equation,

$$dX(u) = \beta\big(u, X(u)\big)du + \gamma\big(u, X(u)\big)dW(u), \tag{6.9.46}$$

as in Exercise 6.8, use the same notation $p(t, T, x, y)$ for the transition density, and again assume that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$. In this problem, we show that $p(t, T, x, y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial T}p(t, T, x, y) = -\frac{\partial}{\partial y}\big(\beta(t, y)p(t, T, x, y)\big) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\big(\gamma^2(T, y)p(t, T, x, y)\big). \tag{6.9.47}$$

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Exercise 6.8 (Kolmogorov backward equation)**.**

Consider the stochastic differential equation

$$dX(u) = \beta\big(u, X(u)\big)du + \gamma\big(u, X(u)\big)dW(u).$$

We assume that, just as with a geometric Brownian motion, if we begin a process at an arbitrary initial positive value $X(t) = x$ at an arbitrary initial time t and evolve it forward using this equation, its value at each time $T > t$ could be any positive number but cannot be less than or equal to zero. For $0 \leq t < T$, let $p(t, T, x, y)$ be the transition density for the solution to this equation (i.e., if we solve the equation with the initial condition $X(t) = x$, then the random variable $X(T)$ has density $p(t, T, x, y)$ in the y variable). We are assuming that $p(t, T, x, y) = 0$ for $0 \leq t < T$ and $y \leq 0$.

Show that $p(t, T; x, y)$ satisfies the *Kolmogorov backward equation*

$$-p_t(t, T, x, y) = \beta(t, x)p_x(t, T, x, y) + \frac{1}{2}\gamma^2(t, x)p_{xx}(t, T, x, y). \qquad (6.9.43)$$

Deep Learning of Transition Probability Densities for Stochastic Asset Models with Applications in Option Pricing

Theorem 6.4.1 (Feynman-Kac). *Consider the stochastic differential equation*

$$dX(u) = \beta\big(u, X(u)\big)du + \gamma\big(u, X(u)\big)dW(u). \qquad \textit{(6.2.1)}$$

Let $h(y)$ *be a Borel-measurable function. Fix* $T > 0$, *and let* $t \in [0, T]$ *be given. Define the function*

$$g(t, x) = \mathbb{E}^{t,x}h(X(T)). \qquad \textit{(6.3.1)}$$

(We assume that $\mathbb{E}^{t,x}|h(X(T))| < \infty$ *for all* t *and* x*.) Then* $g(t, x)$ *satisfies the partial differential equation*

$$g_t(t, x) + \beta(t, x)g_x(t, x) + \frac{1}{2}\gamma^2(t, x)g_{xx}(t, x) = 0 \qquad (6.4.1)$$

and the terminal condition

$$g(T, x) = h(x) \qquad \textit{for all } x. \qquad (6.4.2)$$